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A Simple Approach for Shallow-Water Solitary Wave Interactions

Paul H. Taylor[1, 2]

[1]Department of Engineering Science, University of Oxford, Oxford, OX1 3PJ, UK
[2]Faculty of Engineering, Computing and Mathematics, University of Western Australia,
Crawley, WA 6009, Australia

Abstract

This paper presents analysis of the nonlinear interaction of a pair of solitary waves, the splitting of an isolated hump into a train of solitary waves, and the fission of a single solitary wave on reaching a shelf. Such interactions and splitting behaviour are usually described in the Korteweg–de Vries equation using the Hirota method, or more generally in terms of inverse scattering, a type of nonlinear Fourier transform theory. However, virtually identical results can be obtained using simple physical arguments based on a combination of mass, momentum and energy conservation, with linear scattering from the shelf edge for the shelf interaction. This approximate approach can also be applied to other hydrodynamic equations such as the regularised long wave or Peregrine equation for which inverse scattering methods are not possible.

Introduction

In the mid 19th century John Scott Russell first studied shallow water solitary waves experimentally and noted their persistence through nonlinear interactions. Both Boussinesq and Rayleigh demonstrated mathematically the existence of steady solitary waves on shallow water before Korteweg and de Vries published their famous PDE, which was actually first derived by Boussinesq. After this work the theory of solitary waves remained virtually untouched for 70 years until the mid 1960s when numerical studies by Zabusky and Kruskal [1] revealed the robust nature of soliton interactions, prompting an explosion of sophisticated mathematical analysis on nonlinear PDEs. The history of solitary waves has been reviewed by Miles [2], and that of water waves more generally by Darrigol [3].

The Korteweg–De Vries (KdV) equation describes uni-directional propagation of waves on shallow water, where the horizontal lengthscale of the surface deformation $\eta(x,t)$ is long compared to the undisturbed depth d. In dimensional form this equation is

$$\frac{\partial \eta}{\partial t}+\left(1+\frac{3\eta}{2h}\right)(gh)^{1/2}\frac{\partial \eta}{\partial x}+\left(\frac{h^2}{6}\right)(gh)^{1/2}\frac{\partial^3 \eta}{\partial x^3}=0\,. \qquad (1)$$

There is a solitary wave solution η_{Sol} moving at velocity c

$$\eta_{Sol}=a\,Sech^2\left[\sqrt{3a/4h^3}\,(x-ct)\right] \quad , \quad c=gh\left(1+\frac{a}{2h}\right). \qquad (2)$$

So, solitary waves have a characteristic shape and propagate without change of form, with taller waves being narrower and moving faster. This raises the question of what happens when a faster one overtakes a slower one - a wave-wave interaction first discussed by Scott Russell in the mid 19th century.

Scaling η, x and t, the non-dimensional KdV–equation becomes

$$u_t+(1+u)u_x+u_{xxx}=0\ . \qquad (3)$$

This nonlinear PDE is built up of the following components: the linear telegraph equation $u_t+u_x=0$ corresponds to very long linear waves, the extra triple derivative $+u_{xxx}$ accounts for the dispersion due to finite wavelength, and the $+u\,u_x$ term includes the effect of nonlinearity. Viscous damping is neglected and wave breaking is assumed not to occur.

Assuming we have localised wave disturbances decaying to zero far away, the 1st three conserved quantities of this equation correspond to global conservation of mass, momentum and energy

$$I1=\int_{-\infty}^{\infty}u\,dx\,,\ I2=\int_{-\infty}^{\infty}u^2\,dx\,, I3\ =\int_{-\infty}^{\infty}u^3-3u_x^2\,dx \qquad (4)$$

It is straightforward to prove that these conserved quantities are constants of the motion however complex the wave dynamics. Integrating over the whole spatial domain we can write

$$\int_{-\infty}^{\infty}u_t\,dx\ =-\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left(u\ +\tfrac{1}{2}u^2+u_{xx}\right)dx=0\quad .$$

The wave motion is assumed to decay to zero far away so the RHS is zero. Shifting the time derivative outside the line integral for the LHS, we obtain

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}u\,dx\ =0 \qquad \text{or} \qquad \int_{-\infty}^{\infty}u\,dx\ =I1=\ \text{constant}\,.$$

The higher order conserved quantities can be derived from the PDE similarly. Remarkably the KdV and some other nonlinear evolution equations have an infinite number of such conserved quantities [4], though only the first few of these have a simple physical interpretation. In contrast, other apparently very similar equations, modelling the same physical systems to apparently the same level of approximation, only have a few. The regularised long wave (RLW) equation, also known as the Peregrine or BBM equation [5, 6], only has three conserved quantities [7].

Here we make use of three conserved quantities for the KdV equation, but this analysis could easily be repeated for approximate solutions to the RLW–equation or various versions of the Boussinesq equation.

The nonlinear interaction of a pair of solitary waves

One of the important observations of shallow water wave dynamics reported by Scott Russell in the mid-19th century was what happens when a larger solitary wave overtakes a smaller one, as shown in Figures 1 and 2. He described how the larger one catches up the smaller, then as they start to overlap, mass flows from the larger one behind to the small one ahead. This flux is sufficient to produce a complete exchange of identity, with the larger now ahead of the smaller, so they now separate. In the context of the KdV–equation, the analytic machinery to find exact solutions was only developed in 1968 by Miura et al. [4], and the bi-linear method for soliton interactions was developed slightly later by Hirota [8].

This interaction of a pair of solitary waves can be well approximated using conserved quantities. We assume that the maximally overlapped form shown in the centre of Figure 1 can be modelled by two waves of height A, inverse width B and the soliton $Sech^2$-shape, these being separated by a distance L

$$\eta_{Close} \approx A\,Sech^2\left[B\,x\right] + A\,Sech^2\left[B\,(x-L)\right] \quad .$$

Clearly this assumed shape has 3 undetermined parameters, A, B and L. These can be found in terms of the heights of 2 interacting solitary waves a_1 and a_2 using the 3 conserved quantities $I1$-$I3$. For the two initially well separated solitary waves, we have

$$I1 \sim a_1^{1/2} + a_2^{1/2}, \quad I2 \sim a_1^{3/2} + a_2^{3/2}, \quad I3 \sim a_1^{5/2} + a_2^{5/2} \quad .$$

For the assumed overlap shape, we have $I1 \sim A/B$ and other more complicated but closed form expressions for $I2$ and $I3$ (not given here but obtained using Mathematica). With 3 equations in 3 unknowns, the approximation for the maximally overlapped form can be obtained. The waves shown in Figure 1 are analytical solutions to the KdV–equation, but the approximated overlapped form cannot be distinguished from the exact solution on a plot, the largest error being typically <0.5% of the peak value.

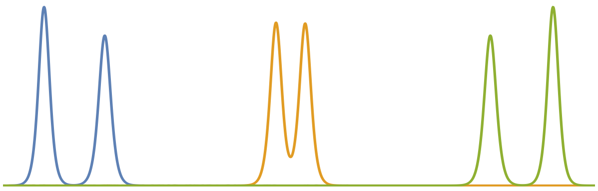

Figure 1. Spatial profiles of overtaking solitary waves, before (at left), at closest approach and after (at right), showing complete restoration of the ingoing waves but with the swap of order.

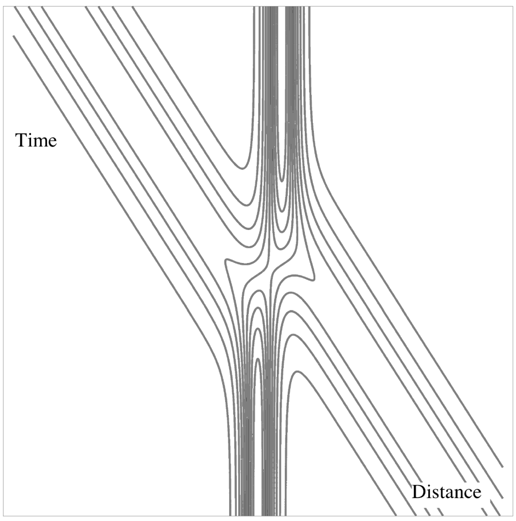

Figure 2. A *(x,t)* contour plot for overtaking solitary waves in a frame of reference moving with the taller wave, showing the exchange of identity and the net forward shift in position for the taller wave and backward for the smaller wave.

We note in passing that this simple model is only valid if the height of the smaller solitary wave is at least one quarter that of the larger, so $a_2 \geq a_1/4$. The overlap length L is reduced to zero for $a_2 = a_1/4$ when perfect merging occurs. For a very small 2nd wave, with $a_2 < a_1/4$, the large wave simply runs over the smaller, although again linear superposition does not apply.

As a generalisation of this analysis, the complete interaction for a pair of solitary waves, such as those shown in Figures 1 and 2, could be modelled with conserved quantities $I1$-$I4$ for the KdV–equation, by generalising the overlapped form to have different amplitudes and widths in each peak, giving 4 dependent parameters. These 4 parameters could then be determined as functions of the separation width L as it is reduced from infinity to the minimum value. Such an analysis would yield a continuous approximation for the spatial structure at all times, not only at the instant of maximum overlap, though not the evolution in time itself as time would remain a hidden parameter.

In contrast to this complete spatial model for a 2-solitary wave interaction in the KdV–equation, no such complete model would be possible for the RLW–equation, despite this equation being an equally valid reduction of the full water waves equations. This is

$$u_t + (1+u)u_x - u_{xxt} = 0 \quad . \tag{4}$$

The only apparently minor difference is the replacement of a single spatial derivative in the dispersion term by one in time. With only 3 conserved quantities, as proved by Olver [7], only an approximation for the maximally overlapped form is possible. The RLW–equation does have solitary wave solutions. But, although numerical solutions to the RLW–equation remain remarkably similar to those for the KdV–equation over long distances and times, these solitary waves are <u>not</u> solitons, generally being very slightly modified by the so-called inelastic interactions [9].

Splitting of an isolated wave hump in the KdV–equation

For the height of the smaller solitary wave being one quarter of that of the larger, so $a_2 = a_1/4$, Figure 3 shows perfect instantaneous merging of the two solitary waves, with the previously defined overlap distance L=0. The merged wave has a height of $a_{Merge}=3a_1/4$. We turn to a generalised version of this problem next.

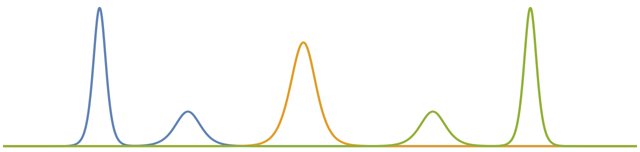

Figure 3. Spatial profiles of overtaking solitary waves, before (at left), at perfect merging and after splitting (at right), showing complete restoration of the ingoing waves but again with the swap of order.

In non-dimensional form a KdV soliton of height a is

$$\eta_{Sol} = a\,Sech^2\left[\sqrt{a/12}\,(x-ct)\right] \quad , \quad c = 1+3a\,. \tag{5}$$

Assuming that n solitons can be combined into a single hump of height A and inverse width B written as

$$\eta_{Merge} = A\,Sech^2\left[B\,x\right] \quad ,$$

the integrals for the conserved quantities can be easily found using Mathematica. For complete merger, these constants of the motion are

$$I1 = 4\sqrt{3}\left(a_1^{1/2} + a_2^{1/2} + \ldots + a_n^{1/2}\right) = 2A/B \quad ,$$

$$I2 = \tfrac{8}{\sqrt{3}}\left(a_1^{3/2} + a_2^{3/2} + \ldots + a_n^{3/2}\right) = 4A^2/3B \quad ,$$

$$I3 = \tfrac{8\sqrt{3}}{5}\left(a_1^{5/2} + a_2^{5/2} + \ldots + a_n^{5/2}\right) = \tfrac{16}{15}A^2\left(A - 3B^2\right)/B \quad .$$

The difficulty now comes in finding solutions to these equations, but a pattern soon emerges if we look for rational solutions. This implies that the a_n-type terms must be simple squares.

We start with n=2 and the solitary wave amplitudes a_1=1 and a_2=1/4 as expected. Substituting and solving for A and B between $I1$ and $I2$ yields the height A=3/4 and the associated inverse width of the merged wave given by B^2=1/48, the perfectly merged case shown in Figure 3. Substitution into $I3$ (and $I4$ and other higher conserved quantities) confirms this solution.

For n=3, the solitary wave amplitudes are a_1=1, a_2=4/9 and a_2=1/9, with the height A=2/3 and the associated inverse width of the merged wave given by B^2=1/108. This 3 soliton case is close to the example of splitting on a shelf shown in Figure 5.

For n=4, the solitary wave amplitudes are a_1=1, a_2=9/16, a_3=4/16 and a_4=1/16, with the height A=5/8 and the associated inverse width of B^2=1/192. Again substitution into $I3$ confirms this solution.

These results are simple to reproduce in numerical solutions of the KdV–equation using NDSOLVE in Mathematica.

The pattern becomes clear: for any n, the soliton amplitudes and parameters of the merged hump are

$$\left[1, \frac{(n-1)^2}{n^2}, \frac{(n-2)^2}{n^2}, \ldots, \frac{1}{n^2}\right] \quad , \quad A = \frac{n+1}{2n} \quad , \quad B^2 = \frac{1}{12n^2}$$

The ratio $A/(12B^2) = A/a$ gives the height of the merged hump A compared to the height a of a solitary wave of the same width. Hence we obtain the general splitting rule: the number of solitary waves emerging from a single $Sech^2$-hump is given by

$$n(n+1) = 2A/a \tag{6}$$

Although this result is well-known and given in Mei [10], the simple ad-hoc derivation given here in terms of conserved quantities is apparently new, though this might be better described as a demonstration as it relies on the identification of the wave pattern noted above.

Solitary Wave Splitting on a Shelf

Consider a solitary wave incident on a step change in water depth as shown below:

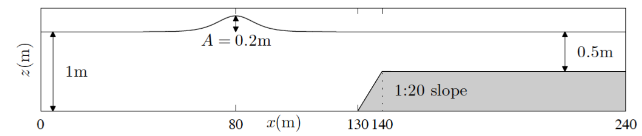

Figure 4. A solitary wave of height 0.2m on water depth of 1m approaching a shelf on which the depth is halved.

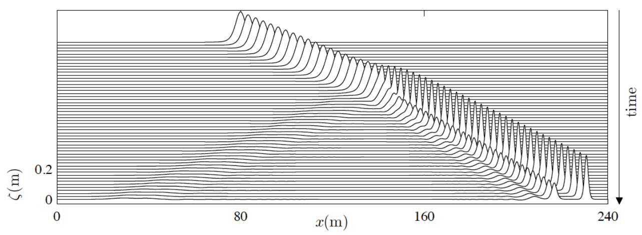

Figure 5. Evolution of a solitary wave up the step, as given in Orszaghova et al. [11] using a Boussinesq model. The single solitary wave splits into 3 solitary waves on the shelf sorted by amplitude, and there is also some weak reflection.

The interaction of a solitary wave with a shelf as computed by Orszaghova using an intermediate complexity Boussinesq model [11] is shown in Figure 5. Rather than continuing with numerical solutions, we now apply analysis based on conserved quantities and linear scattering to this problem. Consider a change in depth, h_1 on the left to h_2 on a shelf on the right ($h_1 > h_2$). A solitary wave approaches the shelf in the deeper region, so there will be partial transmission and then breakup into possibly several solitons on the shelf, and also reflection of a single much smaller pulse back into deeper water.

The nonlinearity in the KdV and other shallow water equations acts slowly and cumulatively, so it is reasonable to treat the rapid interaction with the shelf edge as linear scattering (by analogy to impedance mismatching in transmission lines). Mei [10] gives the transmission (T) and reflection (R) coefficients for a long wave approaching a step change in depth as

$$T = \frac{2}{1+\left(h_2/h_1\right)^{\frac{1}{2}}} \quad , \quad R = \frac{1-\left(h_2/h_1\right)^{\frac{1}{2}}}{1+\left(h_2/h_1\right)^{\frac{1}{2}}}$$

On either side of the step the wave speeds are approximately

$$c_1 \approx \sqrt{gh_1}\,, \qquad c_2 \approx \sqrt{gh_2}\,,$$

The height and length of the disturbance immediately on the shelf determine the number of solitons which then separate out. The timescale for a soliton of height a to pass a fixed location (in a linear approximation) is

$$t \sim 1/c\,\sqrt{4h^3/3a} = \sqrt{4h^2/3ga} \quad .$$

The time between the arrival of leading and trailing edges of the wave does not change as it moves across the shelf edge. So

$$4h_1^2/(3ga_I) = 4h_2^2/3ga_T \,.$$

Thus, a new height $a_T = a_I(h_2/h_1)^2$ is required for the wave on the shelf to be a soliton. The ratio between the actual incident wave height immediately after transmission onto the step and the required height of a single solitary wave of the same duration on the new depth is then

$$\frac{T\,a_1}{a_T} = \frac{2\left(h_1/h_2\right)^2}{1+\left(h_2/h_1\right)^{1/2}} \quad > \quad 1 \;.$$

This has essentially recovered the constant depth splitting problem. We now have a hump of water of amplitude Ta_I on the shallower region, which is too tall to be a soliton a_T matched to the new local depth. Making use of the condition for perfect splitting (6), now on the new reduced constant depth, each critical height ratio for perfect splitting into n solitons is given by

$$n(n+1) = \frac{4\left(h_1/h_2\right)^2}{1+\left(h_2/h_1\right)^{1/2}} \quad . \tag{7}$$

And finding their heights and that of the reflected wave in terms of the incident solitary wave is straightforward.

A second way of treating shelf scattering is to assume that the shelf slope is small, so Green's law is appropriate, see Mei [10]. The slope length is still assumed to be sufficiently compact that nonlinear dynamics on the slope itself are still negligible. Then Green's law gives the transmission coefficient T, with R=0, and the overall splitting criterion becomes

$$T = (h_1/h_2)^{1/4} \quad , \quad n(n+1) = 2\left(h_1/h_2\right)^{9/4} \quad . \tag{8}$$

Johnson used this second approach [12], giving numerical values for the critical depth ratios for n=2 & 3 (to 3 significant figures); see also the equivalent results of Tappert and Zabusky [13]. The calculated values from (7) and (8) are shown in the Table below.

n	h_2/h_1 Eqn (7) - sharp edge	h_2/h_1 Eqn (8) - Green's law
1	1	1
2	0.6116	0.6137
3	0.4470	0.4510
4	0.3541	0.3594
5	0.2940	0.3001

Table: Critical depths for exactly n solitary waves on the shallow region

Johnson then demonstrated the absence of any trailing oscillations at these critical depths using numerical solutions of a variable coefficient KdV–model, assuming a relatively slowly varying transition between the two depths and omitting any weak wave reflection.

For small depth changes in the range $1 > h_2/h_1 > 0.61$, one large solitary wave is produced with a second much smaller one and some small oscillations. For intermediate depth changes with

$0.61 > h_2/h_1 > 0.45$, two large solitary waves are produced with a much smaller third one and some very small trailing oscillations, and so on for larger depth changes. Only at the depth ratio values $h_2/h_1=0.61$ for $n=2$, $h_2/h_1=0.45$ for $n=3$ etc. are the appropriate number of large solitary waves produced with *no* small oscillations trailing behind.

The number of solitary waves produced and the associated critical depths from the gentle slope model are very similar to those for the sharp edged shelf. However, with Green's Law which corresponds to overall conservation of energy into the transmitted wave alone, an important difference is that no wave reflection is predicted for a gentle slope. In contrast, there will be a weak reflected wave for the sharp-edged shelf. This is observed in both physical experiments [14] and the Boussinesq example shown in Figure 5 for $h_2/h_1=0.5$ for a relatively steep slope.

Discussion - the educational value

Much of what is taught in undergraduate engineering courses corresponds to linear theory - the response of systems to small perturbations. But beyond this, there is a whole new world of nonlinearity.

It is striking to observe the reaction of students to a physical demonstration of a larger solitary wave catching a smaller and slower one in a long laboratory flume. Their prior expectations are always of simple linear superposition, with the occasional student suggesting that wave breaking might occur when the taller wave runs over the smaller one. Having done many such demonstrations to 2nd year students in Oxford for over a decade, I have never had anyone suggest beforehand what actually happens – the exchange of identity of the two waves. Having seen it once, students generally don't believe what they've just seen! Hence, repeats are required.

Such a demonstration can be prefaced by an introduction to John Scott Russell, who first observed a solitary wave on the Ardrossan canal outside Edinburgh and then studied solitary waves in a flume constructed in his garden. Scott Russell is perhaps best know as the ship builder who constructed Brunel's ship the *Great Eastern* at Millwall on the banks of the Thames in East London.

It is also perhaps of interest to students to stress the links between the mathematics of long waves on canals (the motivation of the original work by Korteweg and de Vries), the shape of extreme waves on the open ocean [15], and the gigabit transmission rates possible in optical fibres using solitons of light that they may some day use entirely unknowingly when web surfing [16].

Conclusions

A simple approach is presented for solitary wave interactions as modelled with the Korteweg–deVries equation. This makes use of the 3 global quantities that all realistic undamped dynamical systems must conserve: mass, momentum and energy.

The analysis presented here can be applied to a range of nonlinear evolution equations modelling a variety of wave-wave interaction problems: here, solutions to the KdV–equation showing merging and splitting of humps on constant depth, and the splitting of solitary waves as the water depth is reduced. The results are mostly not new, but do show striking effects arising from nonlinearity in one of the simplest possible nonlinear wave equations. Hence, the behaviour of long waves on shallow water would seem ideal for demonstrating to engineering undergraduates that not everything they might study behaves in a linear manner.

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